Dabbling in the Warp: A Brief Journey into Non-Euclidean Geometry

by Casey Davis

Around 2300 years ago, a Greek man named Euclid wrote a book called *Elements*. He began with a rather simple set of definitions and assumptions, and upon this humble foundation he constructed 465 theorems which have become the basis of two- and three-dimensional geometry, as well as much of number theory. Euclid's *Elements* was not the first work of its kind—nor was it by any means the last—but for over two millennia his was *the* definitive treatise on geometry. However, as marvelous as this mathematical wonder is, it is *not* the only possible system of geometry. Various mathematicians in the last two centuries have demonstrated that, while Euclid's geometry is still completely valid under the proper assumptions, other systems, even systems contradicting Euclid, are also viable.

While it is in some senses a work of art (some have called it a "cathedral of geometry," with the postulates as cornerstones), Euclid's system of geometry is above all else a formal system. A formal system consists of *primitive terms*, basic technical terms; *defined terms*, technical terms whose meaning is explained by means of previously introduced terms; *axioms*, primary statements assumed about the terms; and *theorems*, more advanced statements logically deduced from definitions, axioms, and previous theorems (Trudeau 6). Thus (provided that the logic of the system is acceptable) the reliability of the theorems, the final results of the system, can be traced back to the reliability of the axioms and definitions. If we accept these initial conditions and the system of logic used, then we must, logically, accept the theorems deduced from them. If, then, the goal of a formal system is to convince an audience of the truth of its theorems, it would be wise to begin with terms and axioms that are simple and universally accepted. Euclid did just this at the beginning of Book I of his *Elements*: his 23 primitive and defined terms merely explain what he means by such concepts as "point," "line," "circle," "right angle," and so on (see Appendix A for a complete list of Euclid's definitions). His axioms consist of five "common notions," which are simple algebraic and geometric concepts concerning equality of figures and/or numbers and their combinations, and five postulates, assumed statements about the existence and state of straight lines, circles, right angles, and nonparallel lines (see Appendix B for a complete list of Euclid's common notions and postulates). These initial statements—with the possible exception of the controversial fifth postulate, which will be discussed further below—are so simple and elegant that the theorems founded upon them were almost universally accepted throughout the western world as the apex of geometrical truth for over two thousand years.

Despite the complaints of countless thousands of high school students, Euclid's *Elements* really is a very elegant work. It is organized as a sort of a network of geometrical proofs, beginning with a few simple ones (equilateral triangles exist, based on the axioms and definitions, then progressing to more and more complex proofs, each based on the conclusions of the previous ones. The system follows the proof network format so well that the interdependencies of theorems can even be mapped out as a network diagram, as in the beginning of Lewis Carroll's play *Euclid and his Modern Rivals* (Carroll 8) (see Appendix C for a copy of Carroll's network diagram). For example, Theorem 9 depends on Theorem 8, which depends on Theorem 7, which depends on Theorem 5, which depends on both Theorem 4 and Theorem 3; Theorem 3 is based on Theorem 2, which is based on Theorem 1. All of these theorems depend upon the conclusions of previous theorems, except for Theorem 1 and Theorem 4, which are based solely upon the definitions, postulates, and common notions (Carroll 8; Euclid Book I). So we see that, as in all formal systems, all of Euclid's 465 theorems can trace their validity back to his axioms.

But are the axioms really valid? This question has plagued geometers for over two thousand years, and it is very important, for if the foundational axioms are removed, Euclid's entire "cathedral of geometry" collapses in on itself. The Fifth Postulate, regarding nonparallel lines (see Appendix B) has been particularly controversial, as it lacks the elegant simplicity of the other four postulates and does not seem so obvious, sounding more like a theorem than an axiom (Trudeau 118). Many mathematical historians have even suggested that Euclid himself was uncomfortable with the Fifth Postulate; for the first 28 Theorems in Book I of the *Elements* do not require its invocation, while the remaining 20 in that book (except for Theorem 31) are based on its validity, almost as if Euclid was putting of its use as long as possible (Trudeau 44). But whether Euclid liked it or not, the Fifth Postulate was the most controversial portion of the *Elements*, and for millennia avant-garde mathematicians tried to prove it as a *theorem* using "neutral geometry;" that is, Euclid's definitions, common notions, and the four other postulates. Most of their attempts involved replacing the Fifth Postulate with a similar but simpler postulate, usually regarding parallel lines. The first of these that we know of was a Greek philosopher-scientist named Poseidonios, who was active around two centuries after Euclid wrote the *Elements*. He tried to prove Euclid's Fifth Postulate by replacing it with the postulate "Parallel straight lines are equidistant" (Trudeau 119-120). Using this new postulate, together with the context of neutral geometry, it *is* possible to prove the Fifth Postulate; however, it is also possible to prove Poseidonios's Postulate using Euclid's Fifth Postulate together with the context of neutral geometry, so the two theorems are logically identical, and this circular logic has not really accomplished anything. Dozens of later mathematicians tried to succeed where Poseidonios could not, creating many postulates to replace Euclid's Fifth Postulate—some not directly mentioning parallel lines at all—but all failed as he did (Trudeau 126-131). The primary reason for this is that while some of the replacement postulates *seemed* radically different from Euclid's Fifth Postulate, they all were *logically* equivalent to it, and therefore failed in the same area where it did.

By the beginning of the 19th century, after trying and failing to prove Euclid's Fifth Postulate, several prominent mathematicians had actually begun to seriously consider the possibility that such a proof was not only difficult to find, but in fact *impossible*. This would mean that Postulate Five is *not* implied by neutral geometry alone, and in fact “is supported solely by the judgment of our senses” (Trudeau 154). Of course, our senses are close to meaningless in geometry because they lack the infinite precision required, for example, to distinguish between two lines that are parallel and two lines that are *almost* parallel, but meet after, say, 10 000 light-years. A fine and nearly immeasurable distinction, but in geometry a very crucial one. But in any event, the conclusion—not proven, but inferred—was finally reached that “neutral geometry by itself does not imply Postulate 5” (Trudeau 154).

A few decades later, some mathematicians—notably Gauss, Schweikart, Bolyai, and Lobachevsky—began to make the transition from “neutral geometry by itself does not imply Postulate 5” to “a new geometry contrary to Euclid’s is logically possible” (Trudeau 154). Specifically, they removed the Fifth Postulate and replaced it with a new postulate (called ~Playfair’s Postulate; that is, the opposite of Playfair’s Postulate) which was *not* logically equivalent to Euclid’s Fifth Postulate, but in fact its logical *opposite*: that, given a straight line and a point not on that line, there exist not only one, but *multiple* straight lines through the point that are parallel to the given line (Trudeau 159)! Using this new postulate, plus neutral geometry, these new geometers rejected all theorems depending on the Fifth Postulate and worked out an entirely new set of theorems based on this new “hyperbolic geometry” (from Greek *hyperbole*, excess; that is, an excess of parallels). By demonstrating that hyperbolic geometry was as consistent (that is, noncontradictory) as Euclidean geometry, they showed that a system containing neutral geometry plus the negation of Euclid's Fifth Postulate and proved once and for all that Euclid's Fifth Postulate does not automatically follow from neutral geometry.

Some interesting other conclusions were drawn from hyperbolic geometry; that is, using Euclid's primitive terms, definitions, common notions, and first four postulates, along with an extension of ~Playfair's Postulate that we shall call the Hyperbolic Postulate. First of all, Euclid's Theorems 1 through 28 and 31 can still be considered valid, since they do not depend on the Fifth Postulate (Trudeau 98-99). However, everything based on the Fifth Postulate in any way must be rejected. This includes such seemingly obvious statements such as "lines parallel to the same line are parallel," "the sum of a triangle's angles is 180°," a formula for the area of a parallelogram, a proof for the existence of squares, and the Pythagorean Theorem (Trudeau 99). These are replaced by an interesting network of completely different theorems. For example, in hyperbolic geometry (neutral geometry plus the Hyperbolic Postulate), it can be proven that, given a straight line AB and a point P above it, there are two "asymptotic parallels" through P, one sloping down to the left and the other sloping down to the right, that asymptotically *approach* AB, but never actually reach it, thus fitting Euclid's definition of "parallel." Additionally, there are an infinite number of "divergent parallels" through P and above the two asymptotic parallels that not only never meet AB, but constantly "curve" *away* from it (Trudeau 178-179). Conversely, any lines through P and under the asymptotic parallels meet AB at some point and are therefore not parallel to it. Certain new figures are also allowed, including "biangles," which appear similar to triangles but extend infinitely in the direction where one of the corners ought to be because two of the sides are asymptotic parallels, and never actually meet at a point (199). In Euclidean geometry this would not be considered a figure at all, since two of its sides, being parallel, always remain equidistant. The Saccheri quadrilateral (a certain sort of four-sided figure, with two sides perpendicular to the base and equal to each other), which is equivalent to a rectangle in Euclidean geometry, is also very different in hyperbolic geometry (Trudeau 132-147).

We have seen some of the radical differences resulting from negating Euclid's Fifth Postulate in hyperbolic geometry. But is this the only change that can be made to the system? Perhaps not. In 1854, George Riemann delivered a paper claiming that "replacing Postulate 5 with its negation was not the *only* way Euclidean geometry could be tampered with, and within a few years other consistent non-Euclidean geometries made their appearance" (Trudeau 158). One of the most interesting of these, called Riemann double-elliptic geometry, involves redefining a maximally extended straight line as *boundless*, but *finite* in length. This implies that the continued straight line somehow curves back on itself and connects to its other end if it is extended far enough. This negates both the Second Postulate, which implies that a straight line can be extended infinitely; and the First Postulate,[1] which implies that between any two points there can be one and only one straight line. Neither of these are true if an extended straight line eventually "curves" back on itself, because then the line *cannot* be extended infinitely, and two opposite points on the "loop" would then have at least *two* possible straight lines between them (one "curving" in each direction). Interestingly, this new geometry does *not* negate the Fifth Postulate, which merely says that certain pairs of lines are not parallel. Rather, it *extends* the Fifth Postulate in a theorem proving that *no* lines are parallel! Double-elliptic geometry instead negates Theorem 27 of the *Elements* (Euclid Book I), which proved that parallel lines can exist under certain conditions. This proof depends on the "one and only one" implication of the First Postulate, and is therefore invalid in double-elliptic geometry. Furthermore, in double-elliptic geometry it can be proved that all lines perpendicular to a given line meet at a certain point, that all maximally extended straight lines have the same finite length, that two given maximally extended straight lines must meet in *two* points, not one, and that the sum of the angles of a triangle is not fixed, is always greater than 180°, and increases with the size of the triangle[2] (Kline 86)! Even π (that is, the ratio of a circle's circumference to its diameter) is not a fixed constant in double-elliptic geometry, and in fact varies with the size of the circle, much in the same way that the angle-sum

[1] Actually, the First Postulate just says "To draw a straight line from any point to any point," but in many theorems it is clear that Euclid used it to mean "one and only one straight line."

[2] This last point has the interesting consequence that a triangle in double-elliptic geometry can have three right angles.

varies with the size of a triangle.

By this point we have seen that much of the main body of a geometrical system can be changed merely by changing one or two of the initial axioms. Now this is all very nice logically, but what does it really *mean*? In terms of the formal system alone, the answer to this is “nothing.” A true formal system assigns no real meaning at all to its primitive terms; it is merely a system of logical connections between meaningless symbols. Any meaning assigned to these symbols and their interrelationships must be in the domain of the metasystem a separate, non-formal system that looks for deeper meaning behind the formal system. In geometry, the formal system consists of primitive terms, definitions, axioms, theorems, and the logical proofs that link them together: nothing more. Even the diagrams are not truly parts of the proofs, but merely visual aids to help the reader follow the proofs. However, *meta*geometry would say that although geometry is a formal system, it is supposed to *represent* real shapes, lines, angles, and so on, whether they are in the physical world or some Platonic “world of ideas,” and that the diagrams that accompany the proofs are illustrations of that reality. Euclidean geometry is, of course, the geometry of a flat undistorted plane. In a plane without curvature or distortion, all of Euclid’s postulates are completely true: through any two points there is one and only one straight line, a straight line can be continued infinitely, a given center and radius can produce one and only one circle, all right angles are equal to each other, and any given line has exactly one parallel line through a given point not on that line. However, when the plane is curved or distorted in certain ways, some of these postulates are no longer valid. This is, according to metageometry, what non-Euclidean geometry represents: a *curved or distorted* plane. For example, imagine a circle in which length is distorted between the center and the circumference in such a way that if you began walking from the center out towards the circumference, you would shrink accordingly; thus the circle would seem infinitely large, since you could never reach the circumference. And of course it would be impossible to directly notice this distortion, because any measuring devices you brought with you would shrink proportionately (Trudeau 236-238). Now if we use this circle as our plane (this is called Poincaré's Model), and assume that "straight line" means "the shortest distance between two points," then any straight line must be, due to distortion, a segment of a circle that intersects the circumference of our "plane" at right angles. The only straight lines that would seem "straight" to somebody outside the plane, then, would be ones going through the center of the plane. Now with a little work, it can be demonstrated that this system does *not* follow Euclid's Fifth Postulate, but instead follows the Hyperbolic Postulate! It is possible to have an infinite amount of straight lines parallel to a given straight line and through a given point not on that line, since these straight lines "curve" away from each other (Fletcher). Of course, relative to the plane, they are all straight, since their distortion is exactly the same as that of the plane. Thus Poincaré's Model is represented perfectly by hyperbolic geometry. The outer surface of a trumpet's bell, when treated as a plane, also operates under hyperbolic geometry (Trudeau 171).

And what about double-elliptic geometry? The best visual representation of this is probably a plane curved into the shape of a sphere. In this spherical plane, a straight line between two points would be a segment of a "great circle" (Kline 86), or the largest possible circle through those two points. This explains the invalidity of the Second Postulate in double-elliptic geometry: a straight line cannot be continued indefinitely, for after going all the way around the plane it will loop back on itself, at which point it can go no further, being unbounded but finite. The First Postulate (or at least the "one and only one line" implication of it) is also violated, since between two antipodal (opposite) points, any number of straight lines can be drawn. The triangle with three right angles that I mentioned earlier is simple to draw: on a globe, for example, it would have one corner on the North Pole and the other two on the equator, 1/4 of the circumference apart from each other. And of course, there could be no parallel lines, since all straight lines, being great circles, must intersect somewhere. In fact, any two straight lines will intersect at two antipodal points.

So what are the implications of all this? Some geometrical "truths" that we take for granted are not necessarily true in all geometrical systems. We assume that pi is a constant, and yet on a spherical plane it can vary from its standard value all the way down to zero, decreasing as the diameter of the circle increases.[3] We assume that the angles of a triangle always sum

[3] It can even be greater than its standard value on a radially wrinkled plane (such as, for example, that described by the graph of $z=r\bullet\cos[5\theta]$).

to 180°, but in spherical geometry that, too, is variable. Even the concepts about parallel lines that mathematicians have tried to prove for the last two millennia do not hold up in hyperbolic geometry.

Until rather recently, it was assumed that, even though other geometrical systems were *possible*, our universe was definitely Euclidean. But even that has come under question. The Earth itself can be considered a non-Euclidean geometrical system: its surface (disregarding hills, valleys, oceans, and other nonuniformities) can be treated as a spherical plane. Lines of longitude would be examples of straight lines,[4] meeting at the antipodal North Pole and South Pole. This is why planes seem to travel in curves when depicted on flat maps: on the spherical plane, their path really *is* straight, i.e., part of a great circle; it only *seems* curved when distorted onto a flat map. Even the universe itself might not really be Euclidean, but rather a three-dimensional non-Euclidean system: perhaps either inside a spherical version of Poincaré's circle, billions of light-years in diameter (Trudeau 243-244), or bent through a fourth dimension into a sort of a hypersphere (Kline 86).

Of course, it is true that our universe *seems* Euclidean when we perform everyday geometrical measurements. This is actually because *any* plane, regardless of curves, distortions, or nonuniformity, is *almost* flat on a very small level, much in the same sense that the Earth seems flat on a small scale. In fact, our everyday measurements are on such a small scale

[4] Lines of latitude are actually not straight lines at all, except for the equator. Ironically, on most maps, latitudes are shown as straight lines while longitudes are not.

compared to the curvature/distortion/nonuniformity of the universe (if there is any) that on this local level, Euclidean geometry is a *very* close approximation of reality: parallel lines remain equidistant as close as we can measure them, the angle-sum of a triangle differs negligibly from 180°, and π is extremely close to 3.141592...[5] . However, on a much larger scale (e.g., long-distance space travel or mapping the entire Earth), the differences between Euclidean geometry and reality become apparent, and if we stubbornly continue to persist in using Euclid's conclusions, hideous errors could show up, such as the Mercator Projection and crashing into the sun.

These concepts have some interesting implications about the nature of geometrical truth, and in fact truth in general. There is no single *absolute truth* or true geometrical system; all "truths" can only be considered true relative to their respective axiom systems. We cannot simply say, "This geometrical system is right and the others are wrong." Even if we could measure angles and lengths precisely enough to know for sure what system of geometry is the basis of this universe, that is no guarantee that other geometrical systems cannot be true, since they are all theoretical anyway. The truth of a system cannot really be known: the only hint of truth that we can actually find is *internal consistency*. Although we cannot know truth, we can say that a system is valid if it remains consistent. "There are no diamonds [of absolute truth]. People make up stories about what they experience. Stories that catch on are called 'true'" (Trudeau 256). Consistency, then, is the only measure of the "truth" of an explanation.

[5] If you really need more numbers, ask Matt Godwin.

Appendix A: Euclid's Definitions
(from Euclid's *Elements*, Book I)

Primitive Terms:

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A *plane surface* is a surface which lies evenly with the straight lines of itself.

Defined Terms:

8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called *rectilinear*.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.
11. An *obtuse* angle is an angle greater than a right angle.
12. An *acute* angle is an angle less than a right angle.
13. A *boundary* is that which is an extremity of anything.
14. A *figure* is that which is contained by any boundary or boundaries.
15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it [the *radii*] from one point among those lying within are equal to one another.
16. And the point is called the *center* of the circle.
17. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.
19. *Rectilinear* figures are those which are contained by straight lines, *trilateral* figures [*triangles*], being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.
20. Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides… equal, and a *scalene triangle* that which has its three sides unequal.
21. Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.
22. Of quadrilateral figures, a *square* is that which is both equilateral and right-angled…
23. *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Appendix B: Euclid's Common Notions and Postulates
(from Euclid's *Elements*, Book I)

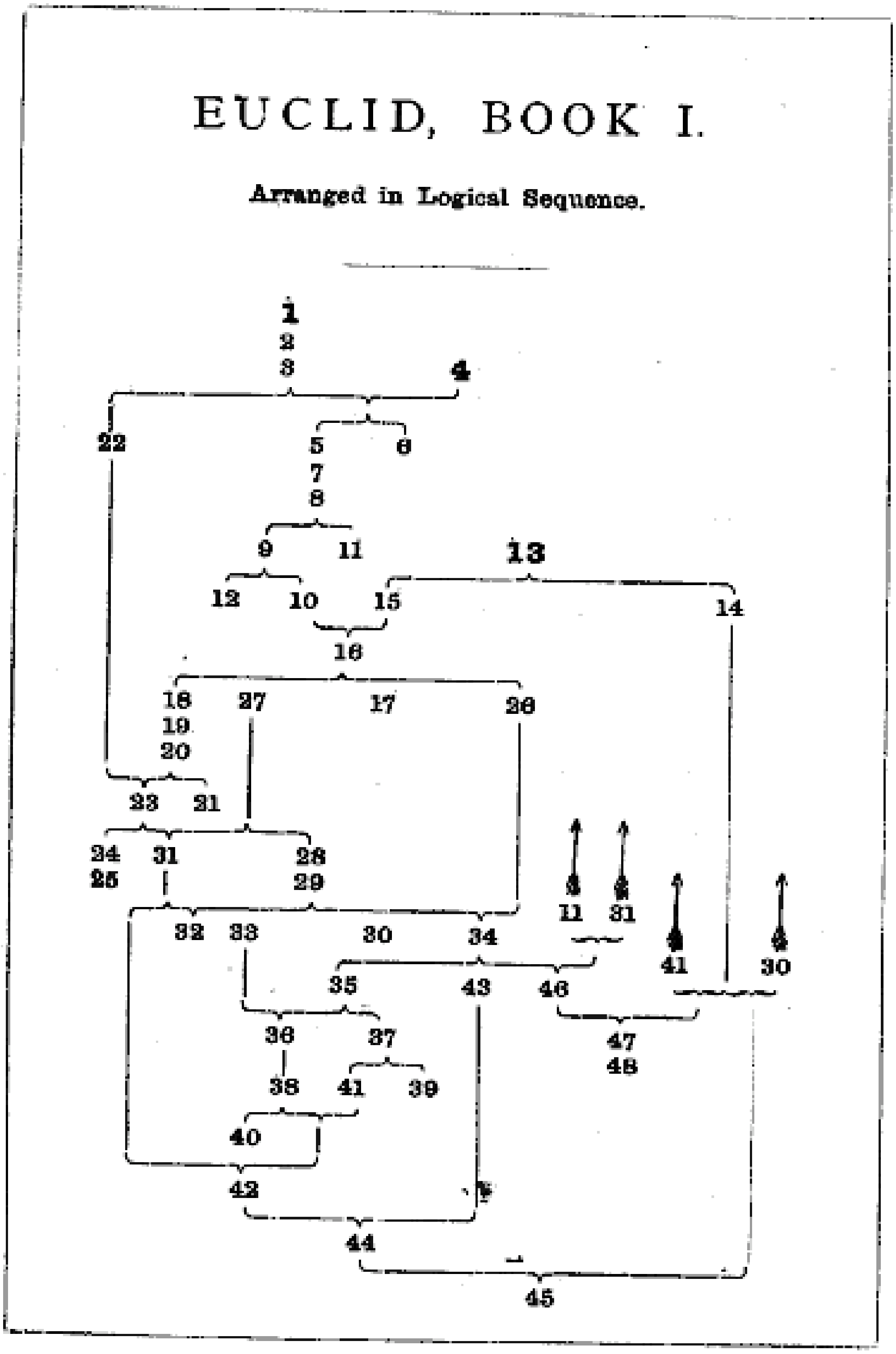
Common Notions:

1. Things which are equal to the same thing are also equal to one another.
   [If a=b and b=c then a=c.]
2. If equals be added to equals, the wholes are equal.
   [If a=b and c=d then a+c=b+d.]
3. If equals be subtracted from equals, the remainders are equal.
   [If a=b and c=d then a-c=b-d.]
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.
   [a+b>a.]

Postulates:

1. [It is possible] to draw [one and only one] straight line from any point to any point.
2. [It is possible] to produce a finite straight line continuously in a straight line.
3. [It is possible] to describe [one and only one] circle with any center and [radius].
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Appendix C: Lewis Carroll's Network Diagram
(from Carroll's *Euclid and his Modern Rivals*)



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