MAT 22B, Fall 22 – Final – Practice

Instructions:

- Unless otherwise stated, **justify all of your answers**. Partial credit will be given to partially correct answers. Answers with no justifications might not receive credit at all.
- Write all answers clearly in the provided space. Use reverse side of pages if needed.
- A scientific calculator is not allowed. It is not needed, you do not need to evaluate expressions of a form $\log(15)$.
- Textbooks, notes, and your own scratch paper are not allowed.

Name: _____ **Id**: _____

Name:

Problem 1: Solve the initial value problem

$$y'' + 2y' + 10y = 0, \quad y(0) = 1,\, y'(0) = -1.$$

Name:

Problem 2: Solve the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Name:

Problem 3: A $10kg$ object is suspended on a vertically hanging spring with $k = 90kgs^{-2}$.

(a) What is the frequency of oscillations if the initial displacement is

i) $y(0) = 0.1m$

ii) $y(0) = 0.05m$

iii) $y(0) = 0.2m$?

(b) At time $t = 0$, the spring-mass system is disturbed from its equilibrium state by a force $F_a(t)$. This experiment is repeated four times with four different forces (all forces are in the units $kgms^{-2}$),

A) $F_a(t) = 10\cos(2t)$ B) $F_a(t) = 10\cos(3t)$

C) $F_a(t) = 100\sin(2t)$ D) $F_a(t) = 100\sin(4t)$.

State the initial value problem describing the objects's motion for the force A) above.

(c) Which of the above forces should we pick if we want to maximise the displacement of the object from its equilibrium position?

Name:

Problem 4: Consider the following four systems of differential equations

A)

$$y_1' = -y_1$$
$$y_2' = -y_2$$

B)

$$y_1' = y_2$$
$$y_2' = y_1$$

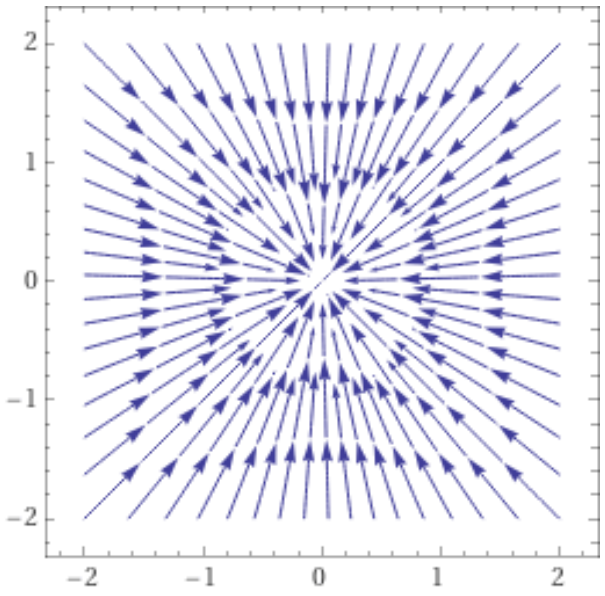
C)

$$y_1' = -y_2$$
$$y_2' = y_1$$

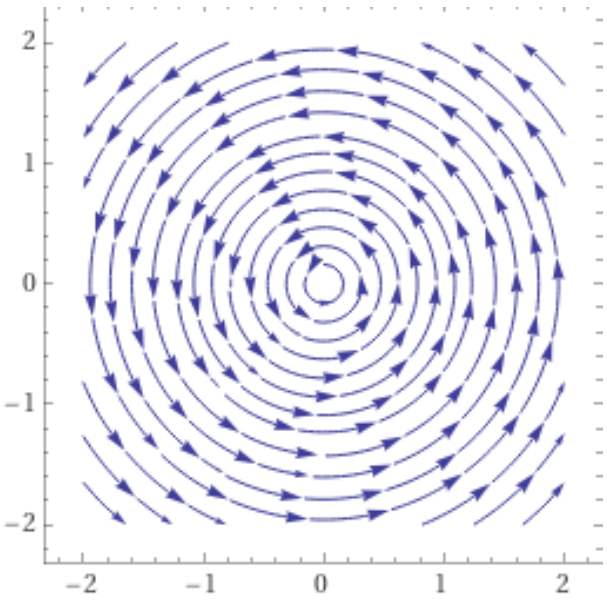
D)

$$y_1' = y_2$$
$$y_2' = -y_1$$

a) Match the equations A)-D) to the following phase-space pictures

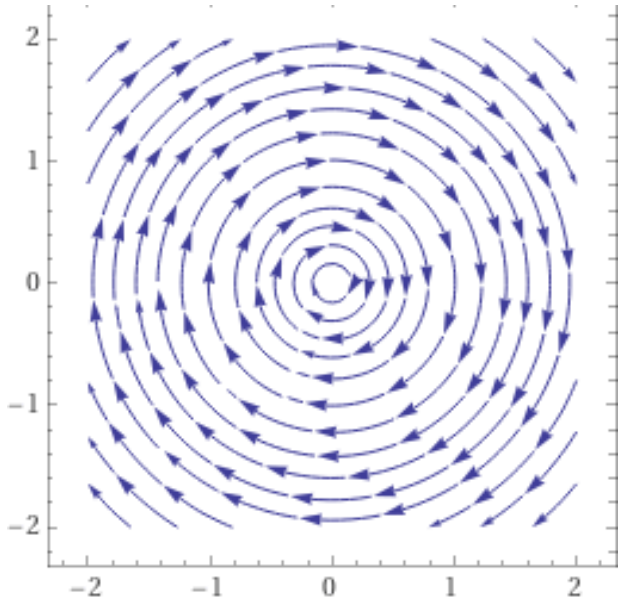


1

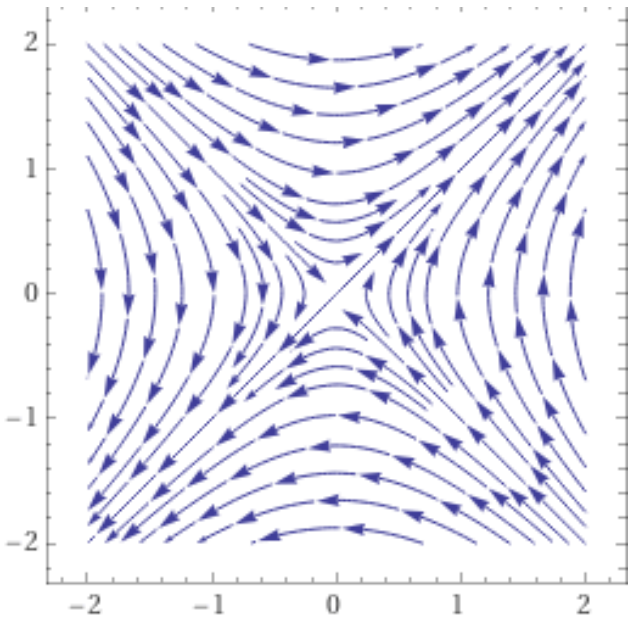


2

Name:



3



4

b) Consider the initial conditions $y_1(0) = -1, y_2(0) = 1$. For each of the systems $A) - D)$ decide if the corresponding solution $y_1(t), y_2(t)$ stays confined or escapes to infinity.

Name:

Problem 5. Radioactive decay is a process in which an unstable atom transmutes over time into a stable chemical element. The law of radioactive decay says that the number of unstable atoms (atoms that did not decay yet), $N(t)$, depends on time t via the equation

$$\frac{dN}{dt} = -\lambda N,$$

where λ is the decay rate.

For a particular atom, engineers observed, that after 100 days, only 10 grams of the radioactive substance was left out of an initial 30 grams.

(a) Determine the decay rate λ.

(b) Find the time t for which the amount of the radioactive substance is equal to the half of the original amount.

(c) Write one reason why this is a good model, and one reason why this is a bad model.

Name:

Problem 6. A researcher was studying a small forest with population of foxes and rabbits. After several years of observations they concluded that the population of rabbits oscillate around 1000 rabbits and the population of foxes oscillates around 100 foxes. To model that they wrote the population of rabbits as $P_1 = 1000 + y_1$ and the population of foxes as $P_2 = 100 + y_2$.

(a) Pick which of the following four differential equations for y_1, y_2 best models the above situation.

$$A),\ \mathbf{y}' = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \mathbf{y}, \quad B)\ \mathbf{y}' = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \mathbf{y}$$

$$C),\ \mathbf{y}' = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix} \mathbf{y}, \quad D)\ \mathbf{y}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}$$

(b) For equation B) and initial condition $y_1(0) = 0, y_2(0) = 10$, what is the maximal amount of rabbits predicted by this model over time?

(c) Which of the following statement is correct prediction based on equation B)

(i) Populations of rabbits and foxes reach maximum at the same time.

(ii) Population of rabbits is maximal when population of foxes is minimal.

(iii) When population of rabbits is maximal the population of foxes is 100.

(iv) None of the above.

(d) What would happen with populations if they evolve according to equation A)?

Name:

That is all. This is an empty page.