Math 22A: Linear Algebra Practice Final

1) Let $A = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 3 & 3 & 4 & 5 & 3 \\ 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Find bases for A's row space, column space, and null space.

$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$C(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}$

$N(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

What is the rank of A? 3

What is the nullity of A? 2

What do these add up to, and why? 5, because rank + null = #columns

Without actually finding the left null space of A, what is its dimension, and why?

1, because rank + leftnull = #rows

Choose a vector in R(A) and a vector in N(A), and calculate their dot product.
Why *must* the result be that value?

0, because R(A) & N(A) are orthogonal

Find the complete solution of $A \cdot \vec{x} = \begin{bmatrix} 6 \\ 4 \\ 11 \\ 3 \end{bmatrix}$. $\vec{x} = \begin{bmatrix} 3 \\ 0 \\ -5 \\ 0 \\ 5 \end{bmatrix} + c_1 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

2) Find a line $y = mx + b$ (that is, find m and b) of best fit for this data set:

$A\vec{m} = \vec{y}$

$A^T A \vec{m} = A^T \vec{y}$

$\vec{m} = (A^T A)^{-1} \cdot A^T \vec{y}$

$\vec{m} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} m \\ b \end{bmatrix} \rightarrow \boxed{y = 2x - 2}$

x	y
1	-1
2	0
3	5
4	6

$$\left.\begin{aligned} 1m + 1b &= -1 \\ 2m + 1b &= 0 \\ 3m + 1b &= 5 \\ 4m + 1b &= 6 \end{aligned}\right\} \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{m}} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 5 \\ 6 \end{bmatrix}}_{\vec{y}}$$

3) Let $\vec{u} = \begin{bmatrix} -3 \\ 4 \\ -4 \\ 3 \end{bmatrix}$ and $W = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

Find an orthogonal basis for W. (You can normalize it too if you want practice.)

$$\boxed{\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2/5 \\ -1/5 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

(no change needed) (projection subtracted) (projections subtracted)

To normalize, divide each one by its own magnitude. (Note that $\vec{v}_3$ is already unit length.)

Find the projection of $\vec{u}$ onto W.

Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$. Then $P_W = A(A^T A)^{-1} A^T = \frac{1}{6} \cdot \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 5 & 2 & -1 \\ 0 & 2 & 2 & 2 \\ 0 & -1 & 2 & 5 \end{bmatrix}$

and $\boxed{\text{proj}_W(\vec{u}) = P_W \cdot \vec{u} = \begin{bmatrix} -3 \\ 3/2 \\ 1 \\ 1/2 \end{bmatrix}}$

4) Is the set of all cubics *without* an x^2 term a vector space? Why or why not?

Yes — closure under addition & scalar multiplication

Is the set of all functions that pass through (2,3) a vector space? Why or why not?

No — it doesn't even include the zero function

5) Find all eigenvalues and eigenvectors of $B=\begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. (Bonus: diagonalize B.)

$\boxed{\lambda_1 = 1,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = 2,\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_3 = -1,\ \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$

(Note: order doesn't matter as long as the pairs stay together.)

Diagonalization:

$\boxed{B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$ (there are others too)

6) Factor the matrix $C=\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ into LPU or PLU form (your choice).

Why is it not possible to factor it into just LU form?

If we only add multiples of rows to lower rows, the pivots are out of order, so a permutation of the rows (before or after) is needed.

If you permute before:

$\boxed{C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$

(PLU)

or

If you permute after:

$\boxed{C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$

(LPU)

7) Let $G=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. Evaluate det(G). $\boxed{0}$

Is G invertible? If it is, find det(G^{-1}) without finding G^{-1}. $\boxed{\text{no}}$

Evaluate det(10 G) with as little work as possible. $\boxed{\text{also } 0}$

because $\det(10G) = 10^4 \cdot \det(G) = 10^4 \cdot 0 = 0$

8) Let $H=\begin{bmatrix}3&4\\6&5\end{bmatrix}$. Find the LDU-decomposition of H.

$-2\cdot R_1 \rightarrow \begin{bmatrix}3&4\\0&-3\end{bmatrix} = U, \quad E=\begin{bmatrix}1&0\\-2&1\end{bmatrix} \rightarrow L=E^{-1}=\begin{bmatrix}1&0\\2&1\end{bmatrix}$

so for LU-decomposition, $H=\begin{bmatrix}1&0\\2&1\end{bmatrix}\cdot\begin{bmatrix}3&4\\0&-3\end{bmatrix}$. But we want LDU!

So insert identity matrix: $H=\begin{bmatrix}1&0\\2&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}3&4\\0&-3\end{bmatrix}$

Factor 3 out of top row of U and -3 out of bottom row:

$$\boxed{H=\begin{bmatrix}1&0\\2&1\end{bmatrix}\cdot\begin{bmatrix}3&0\\0&-3\end{bmatrix}\cdot\begin{bmatrix}1&4/3\\0&1\end{bmatrix}}$$

9) Prove that if $A\vec{x}=\vec{0}$ has only the trivial solution, then $A\vec{x}=\vec{b}$ has no more than one solution.

Suppose $A\vec{x}=\vec{b}$ does have two different solutions $\vec{x}_1$ and $\vec{x}_2$.

Then $A\vec{x}_1=\vec{b}$ and $A\vec{x}_2=\vec{b}$.

Then $A\vec{x}_1=A\vec{x}_2$

Then $A\vec{x}_1-A\vec{x}_2=\vec{0}$

Then $A(\vec{x}_1-\vec{x}_2)=\vec{0}$

So $(\vec{x}_1-\vec{x}_2)$ is a nontrivial solution to $A\vec{x}=\vec{0}$. Contradiction!

Thus $A\vec{x}=\vec{b}$ cannot have more than one solution, QED

10) Let T be a transformation matrix that takes vectors in $\mathbf{R}^2$ as input and produces vectors in $\mathbf{R}^3$ as output. Given the following input-output pairs, find matrix T.

$T\cdot\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$T\cdot\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}0\\9\\3\end{bmatrix}$

$T\cdot\begin{bmatrix}2&3\\3&0\end{bmatrix}=\begin{bmatrix}1&0\\1&9\\1&3\end{bmatrix}$

$\hookrightarrow T=\begin{bmatrix}1&0\\1&9\\1&3\end{bmatrix}\begin{pmatrix}2&3\\3&0\end{pmatrix}^{-1}=\begin{bmatrix}1&0\\1&9\\1&3\end{bmatrix}\cdot\begin{bmatrix}0&1/3\\1/3&-2/9\end{bmatrix}$

$$\boxed{T=\begin{bmatrix}0&1/3\\3&-5/3\\1&-1/3\end{bmatrix}}$$